Magic-induced computational separation in entanglement theory

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Joint work with:

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Talk based on: ArXiv: 2403.19610

MBQM 2024

SCUOLA NORMALE SUPERIORE

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- For states with no magic, estimate entanglement is easy.
- **Operational approach**: let us study the difference for entanglement characterization and manipulation tasks.

Measures of Entanglement and Magic

How do we measure entanglement?

- Bipartition in a qubit system. $A|B$
- Reduced density matrix $\psi_A = \mathrm{tr}_B \, |\psi\rangle\langle\psi|$
- **von Neumann entropy**:

$$S_1(\psi_A) = -\,\mathrm{tr}(\psi_A \log \psi_A)$$

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How do we measure magic?

- Group of Pauli operators $P \in \mathcal{P}_n$
- Pauli subgroup *stabilizing* $|\psi\rangle$

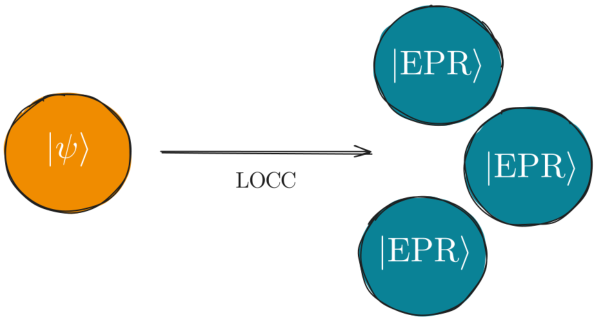
$$G_\psi = \{P : P|\psi\rangle = |\psi\rangle\}$$

- **Stabilizer nullity**

$$\nu(\psi) = n - \log|G_\psi|$$

Entanglement manipulation

Task: via LOCC Alice and Bob want to distill a Bell pair from an entangled state $|\psi\rangle$



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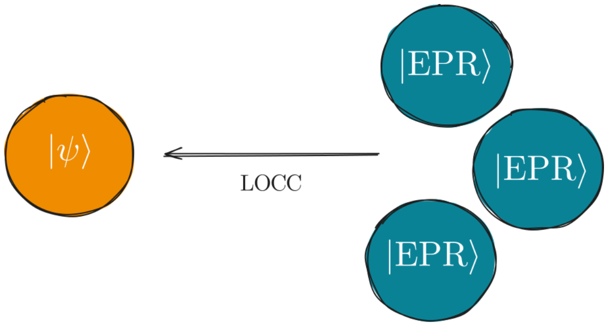

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Stabilizer States

- Consider k mutually commuting, and independent Pauli operators $S = \{P_1, \ldots, P_k\}$.

$$\sigma = \prod_{P \in S} \frac{I + P}{2}$$

- S is the generating set of the stabilizer group G (abelian subgroup of $\mathcal{P}_n$) associated with σ.

$$|G| = 2^{|S|} = 2^k$$

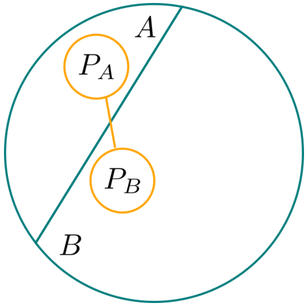
- Pure stabilizer states $|\sigma\rangle \; \forall P \in G$, $P|\sigma\rangle = |\sigma\rangle$.

- All the properties of σ can be determined by looking at S.

Entanglement for stabilizer states

- Entanglement is completely determined by S.
- $S_A = \{P \in S | P = P_A \otimes I_B\}$
- $S_A = \{P \in S | P = I_A \otimes P_B\}$
- $S_{AB} = \{P \in S | P \notin S_A \cup S_B\}$

$$S_1(\sigma_A) = \frac{|S_{AB}|}{2}$$



Example

$$|\text{EPR}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$S_A = \{\}, S_B = \{\}, S_{AB} = \{XX, ZZ\}$$

$$S_1(\text{tr}_B(|\text{EPR}\rangle\langle\text{EPR}|)) = 1$$

Fattal et al., *Entanglement in the stabilizer formalism*, ArXiv:quant-ph/0406168

Magic-States $= \nu$-compressible states

- Consider a state $|\psi\rangle$ with stabilizer nullity ν.
- We can associate a stabilizer group G generated by S.

$$|\psi\rangle\langle\psi| = \sum_{i=1}^{4^{\nu}} \mathrm{tr}(h_i \psi) h_i \prod_{P \in S} \frac{I+P}{2}$$

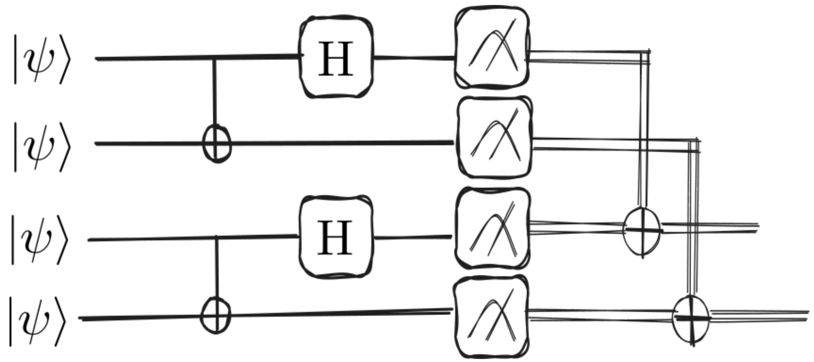
- $|\psi\rangle$ is ν-compressible because it can always be written as $|\psi\rangle = C(|0\rangle_{n-\nu} \otimes |\phi\rangle_{\nu})$
- **Fact:** the stabilizer group G can be learned efficiently.

Leone et al., *Learning t-doped stabilizer states*, Quantum 8, 1361 (2024).

Learning algorithm for G

Algorithm:
Input $O((n+\log(1/\delta))\epsilon)$ copies of $|\psi\rangle$, $\epsilon, \delta \in (0,1)$
Output Stabilizer set $\hat{S}$.

1. Perform Bell difference Sampling. The span of the samples is $S^{\perp}$
2. $S = \mathrm{Ker}(S^{\perp})$ (Gaussian Elimination)



Grewal et al., *Efficient Learning of Quantum States Prepared With Few Non-Clifford Gates*, ArXiv: 23

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- The group $\hat{G} \equiv \langle \hat{S} \rangle$ contains G.
- $|\psi\rangle$ is ϵ-close in trace distance to some state with a stabilizer group $\hat{G}$.
- Runtime $O(n^2(n+\log(1/\delta))\epsilon)$

Grewal et al., *Efficient Learning of Quantum States Prepared With Few Non-Clifford Gates*, ArXiv: 230

Stabilizer Entanglement

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- One can always decompose S on the bipartition A|B: $S = S_A \cup S_B \cup S_{AB}$

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Bounds with entanglement entropy

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- $E(\psi_A)$ can be estimated efficiently $O(n^2)$
- ν can be estimated efficiently $O(n)$

Entanglement vs Magic-dominated

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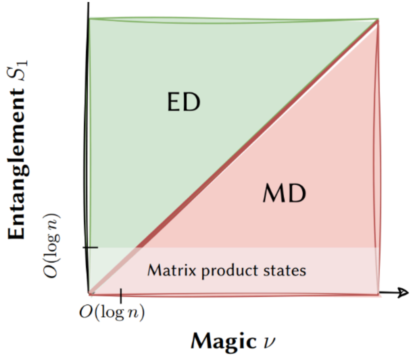

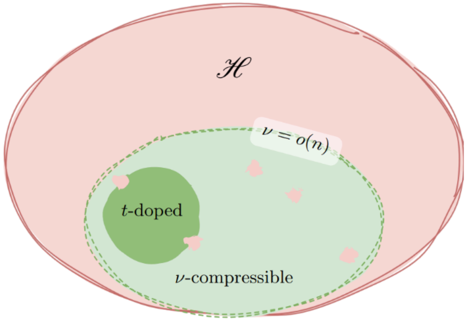

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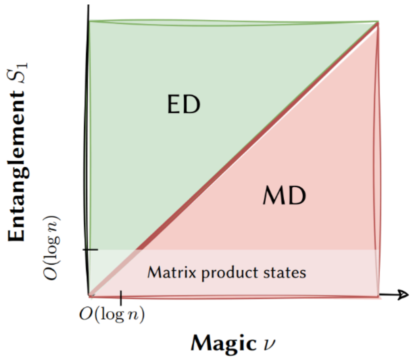

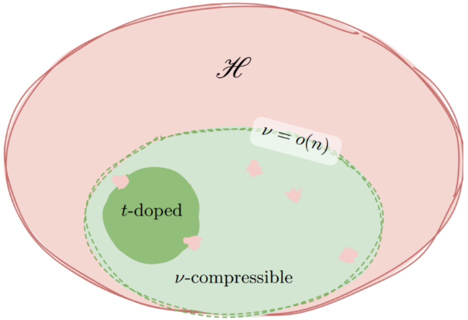

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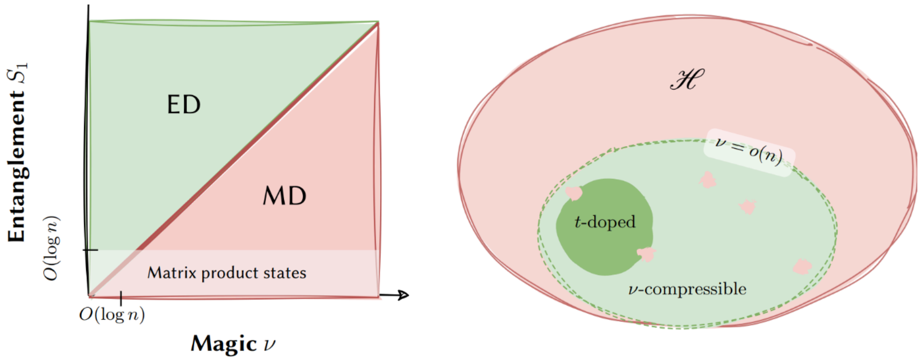

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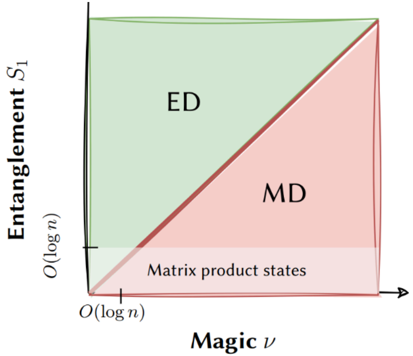

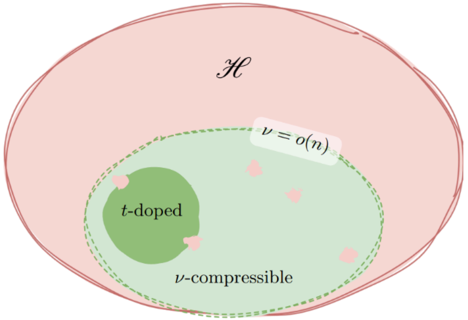

- Matrix Product States (MPS) are magic-dominated but allow potential efficient manipulation.
- ν-compressible states ($\nu = o(n)$) occupy a vanishing fraction of the Hilbert space.
- Entanglement-dominated phase is typical for ν-compressible and t-doped states.

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- **General Case**: Entanglement characterization requires exponentially many measurements for large n_A.
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- **Example**: Volume Law vs Sub-Volume Law
  - If $f(n_A) = n_A \quad \Longrightarrow$ Volume law: $c = \Theta(1)$. Sub-volume law: $c = o(1)$.

Efficient entanglement distillation for entanglement-dominated task

Theorem There exists a bipartite Clifford unitary that distills a number of Bell pair equals to

$$M_+ = E(\psi_A) - \nu/2$$

which, for entanglement dominated states, is asymptotically (in n) optimal: $M_+/S_1(\psi_A) = 1 - o(1)$. Moreover, the unitary, can be found by $O(n)$ queries to $|\psi\rangle$.

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Proof Sketch: $S = S_A \cup S_B \cup S_{AB}$, $|S| \geq n - \nu$

- We can complete the stabilizer group S to a maximal one S^c, describing a stabilizer state $|S^c\rangle$.
- For S^c there exists a unitary $U_A \otimes U_B$ Clifford that distills up to $|S^c_{AB}|/2$ Bell pairs.
- Applying the same unitary on $|\psi\rangle$, it transforms $S \to S'$ obtaining M_+ Bell pairs:

$$M_+ \geq \frac{|S_{AB}| - \nu}{2} = E(\psi_A) - \nu/2$$

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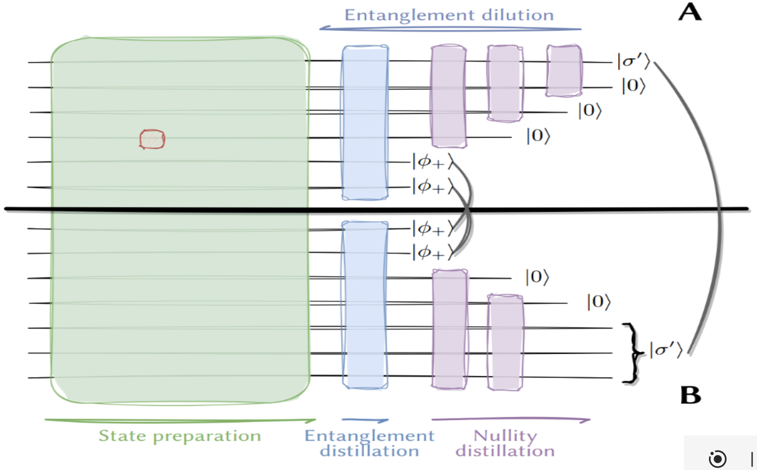
Proof Sketch :

- B runs locally the distillation protocol. Obtaining the state $|\sigma'\rangle$
- Teleport of $\nu/2$ qubits of $|\sigma'\rangle$ to A.
- Application of local Cliffords on A and B. *Equivalent to revert distillation.*



No-go for magic dominated states

Theorem Any efficient state-agnostic protocol that can estimate $S_1(\psi_A)$ within $\omega(1)$ relative error for all MD states. It can distills at most a fraction of $o(1)$ Bell pairs from a magic-dominated state, and diluite more than a fraction of $\omega(1)$ Bell pairs from a magic-dominated state.

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Proof Sketch: Pseudorandom states encoded as magic-states

- Consider the following magic dominated state $|\psi\rangle = |0\rangle_{n-\nu} \otimes |\phi_{AB}\rangle_\nu$, with $\nu = \Theta(\log^c(n))$ with $c > 1$
- Two possible choices either $|\phi_{AB}\rangle_\nu$ is an Haar random state or $|\phi_{AB}\rangle_\nu$ is a pseudo entangled state
- Haar random states have maximal entropy of entanglement $S_1(\phi^H_{AB}) \sim \Theta(\log^c n)$, while $S_1(\phi^P_{AB}) = \Theta(\log^{c'} n)$, an efficient algorithm that achieves $M_+/S_1 = \Omega(1)$ fraction of distillable Bell pairs would distinguish pseudo random states from Haar. Consequently, the maximal number of extractable Bell pairs obeys $M_+/S_1 = o(1)$

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- ε-ED and ε-MD phases
  - $|\psi_\varepsilon\rangle := \mathrm{argmin}\{\nu(|\phi\rangle) \colon \| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1 \leq \varepsilon\}$
  - **ED phase**: $S_1(\psi_{\epsilon A}) = \omega(\nu(|\psi_\epsilon\rangle))$ **MD phase**: $S_1(\psi_{\epsilon A}) = O(\nu(|\psi_\epsilon\rangle))$
- Results on ED-phase and MD-phase can be generalized to this ε-version up to error ε.
- **Proof idea**: Look at the ε-ball, and then use of Fannes inequality.

Future directions

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Thanks for your attention!